

The homomorphism threshold of $\{C_3, C_5\}$ -free graphs

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Abstract

We determine the structure of $\{C_3, C_5\}$ -free graphs with n vertices and minimum degree larger than $n/5$: such graphs are homomorphic to the graph obtained from a $(5k-3)$ -cycle by adding all chords of length $1 \pmod{5}$, for some k . This answers a question of Messuti and Schacht. We deduce that the homomorphism threshold of $\{C_3, C_5\}$ -free graphs is $1/5$, thus answering a question of Oberkampf and Schacht.

1 Introduction

We are interested in the structure of graphs of high minimum degree which forbid specific subgraphs. For a fixed graph H , a graph is said to be H -free if it does not contain H as a subgraph. Let $\text{Forb}(H)$ denote the set of H -free graphs, and let $\text{Forb}_n(H)$ denote the set of n -vertex graphs in $\text{Forb}(H)$. Furthermore, let $\text{Forb}(H, d)$ denote the collection of H -free graphs G with minimum degree at least $d|G|$. Analogous definitions hold if we replace H by some family \mathcal{H} of graphs. Finally, we say that a graph G is homomorphic to a graph H if there exists a map $f : V(G) \mapsto V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. For example, G is homomorphic to K_r if and only if $\chi(G) \leq r$.

A classical result of Andrásfai, Erdős and Sós [3] states that if G is a K_{r+1} -free graph on n vertices with minimum degree $\delta(G) > \frac{3r-4}{3r-1}n$, then G is r -colourable. This result can be viewed as a significant strengthening of the following fact, which is a consequence of Turán's theorem: the minimum degree of a K_{r+1} -free graph on n vertices is at most $(1 - 1/r)n$. Note also here that the chromatic number $\chi(G)$ of G is bounded by a constant independent of n . In general, one may ask whether or not this behaviour persists when the minimum degree condition is weakened. Along these lines, Häggkvist [10] showed that any n -vertex triangle-free graph of minimum degree greater than $3n/8$ is homomorphic to a 5-cycle, and accordingly has chromatic number at most 3.

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Note that this is indeed an extension of the Andrásfai-Erdős-Sós result when the minimum degree condition is weakened, since a balanced blow-up of a 5-cycle exhibits the tightness of that result. Jin [12] took up the investigation and significantly extended the work of Häggkvist: he proved that for all $1 \leq k \leq 9$, any n -vertex triangle-free graph with minimum degree larger than $\frac{k+1}{3k+2}n$ is homomorphic to the graph $F_{k,2}$, which is obtained by adding all chords of length 1(mod 3) to a cycle of length $3k-1$. Observe that $F_{k,2}$ is triangle-free and 3-colourable for every k . The graphs $F_{k,2}$ are a special case of a larger family of graphs, $F_{k,\ell}$, which we shall discuss shortly. We note that Jin's result [12] is best possible, in the sense that such a statement does not hold for $k = 10$. Indeed, by taking a suitable blow-up of the Grötzsch graph (see Figure 1) one can obtain a triangle-free graph on n vertices and minimum degree $\lfloor 10n/29 \rfloor$ which is not 3-colourable, so in particular it is not homomorphic to $F_{l,2}$ for any l . Building on this work, Chen, Jin, and Koh [6] showed, in particular, that any n -vertex 3-colourable triangle-free graph G with $\delta(G) > n/3$ is homomorphic to $F_{k,2}$, for some k . Again, the Grötzsch graph shows that the assumption that the graph is 3-colourable is necessary.

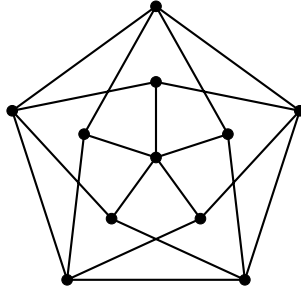


Figure 1: the Grötzsch graph

In general, one may ask for the *smallest* minimum degree condition we may impose on an H -free graph which guarantees that it has bounded chromatic number. To be precise, this prompts us to define the *chromatic threshold* $\delta_\chi(H)$ of a graph H :

$$\delta_\chi(H) = \inf\{d : \text{there exists } C = C(H, d) \text{ such that if } G \in \text{Forb}(H, d), \text{ then } \chi(G) \leq C\}.$$

In other words, $\delta_\chi(H)$ is the least d such that every H -free graph on n vertices and with minimum degree at least dn has bounded chromatic number. This definition was implicit in the works of Andrásfai [2] and Erdős and Simonovits [8], and was first explicitly formulated by Łuczak and Thomassé [15].

For every $\varepsilon > 0$, Hajnal (appearing in [8]) constructed graphs in $\text{Forb}(K_3, 1/3 - \varepsilon)$ with arbitrarily large chromatic number, thereby proving the bound $\delta_\chi(K_3) \geq 1/3$. Thomassen [20] thereafter established the matching upper bound, showing that $\delta_\chi(K_3) = 1/3$. In fact, Brandt and Thomassé [5] strengthened this by showing that triangle-free graphs of minimum degree larger than $n/3$ have chromatic number at most four, answering a question of Erdős and Simonovits [8]. Extensions of these results were obtained by several authors [9, 18], who showed that $\delta_\chi(K_r) = \frac{2r-5}{2r-3}$. Fi-

nally, building off of ideas of Łuczak and Thomassé [15] and Lyle [16], Allen, Böttcher, Griffiths, Kohayakawa and Morris [1] determined the value of $\delta_\chi(H)$ for every graph H with $\chi(H) > 2$.

Note that the results of Häggkvist [10], Jin [12], and Chen, Jin, and Koh [6] mentioned earlier not only show that triangle-free graphs of large enough minimum degree have bounded chromatic number, but that they are actually homomorphic to some specific 3-colourable triangle-free graph. One may ask then, with respect to the above discussion, whether we can replace the property of having bounded chromatic number with the property of admitting a homomorphism to a graph of bounded order with additional properties. This question was posed by Thomassen [20] in the specific case of triangle-free graphs, and motivated Oberkamp and Schacht [19] to introduce the *homomorphism threshold* $\delta_{\text{hom}}(H)$ of a graph H :

$$\delta_{\text{hom}}(H) = \inf\{d : \exists C = C(H, d) \text{ s.t. } \forall G \in \text{Forb}(H, d) \\ \exists G' \in \text{Forb}_C(H) \text{ s.t. } G \text{ is homomorphic to } G'\}.$$

In words, $\delta_{\text{hom}}(H)$ is the least d such that every H -free graph with n vertices and minimum degree at least dn is homomorphic to an H -free graph of bounded order. Note that the definition of $\delta_{\text{hom}}(H)$ extends naturally if we replace H by a family \mathcal{H} of graphs.

Łuczak [14] proved that $\delta_{\text{hom}}(K_3) \leq 1/3$. Note that if G is homomorphic to G' , then $\chi(G) \leq |V(G')|$. Accordingly, we always have $\delta_{\text{hom}}(H) \geq \delta_\chi(H)$, and so, since $\delta_\chi(K_3) = 1/3$, it follows that $\delta_{\text{hom}}(K_3) = 1/3$. This result was extended by Goddard and Lyle [9] to K_r -free graphs for $r \geq 4$, and, in particular, we know that $\delta_{\text{hom}}(K_r) = \delta_\chi(K_r) = \frac{2r-5}{2r-3}$. Oberkamp and Schacht [19] gave a new proof of this result avoiding the Regularity Lemma (which was used in Łuczak's proof), and asked for the determination of the homomorphism threshold of the odd cycle, $\delta_{\text{hom}}(C_{2\ell-1})$, and $\delta_{\text{hom}}(\{C_3, \dots, C_{2\ell-1}\})$ for $\ell \geq 3$. As our first main result, we determine the value of the second of these two parameters in the case $\ell = 3$.

Theorem 1. *The homomorphism threshold of $\{C_3, C_5\}$ is $1/5$.*

In other words, Theorem 1 states that, for every $\varepsilon > 0$, if G is a $\{C_3, C_5\}$ -free graph on n vertices and minimum degree at least $(1/5 + \varepsilon)n$, then G is homomorphic to a $\{C_3, C_5\}$ -free graph of order at most C , where C depends on ε but not on n . We also establish an upper bound on $\delta_{\text{hom}}(C_5)$. This is a consequence of Theorem 1, since C_5 -free graphs of large minimum degree end up being triangle-free as well (see Section 7.1). In particular, we have the following.

Corollary 2. *The homomorphism threshold of C_5 is at most $1/5$.*

In fact, we are able to say much more about the structure of $\{C_3, C_5\}$ -free graphs with n vertices and minimum degree larger than $n/5$. First we need to define a family of graphs. Following Messuti and Schacht [17], for integers $k \geq 1$ and $\ell \geq 2$, denote by $F_{k,\ell}$ the graph obtained from a $((2\ell - 1)(k - 1) + 2)$ -cycle by adding all chords joining vertices at distances $j(2\ell - 1) + 1$ for

$j = 0, 1, \dots, k - 1$. For our purposes, $\ell = 3$ and we shall write F_k instead of $F_{k,3}$ for simplicity. In particular, F_1 is an edge, F_2 is a C_7 (a cycle of length 7) and F_3 is the graph obtained by adding all diagonals to a C_{12} (by a *diagonal* in an even cycle $C_{2\ell}$, $\ell \geq 2$, we mean an edge joining vertices at distance ℓ along the cycle). This graph is also known as the Möbius ladder on 12 vertices (see Figure 2a). It is easy to check that, for each $k \geq 1$, F_k is k -regular, maximal $\{C_3, C_5\}$ -free, and 3-colourable.

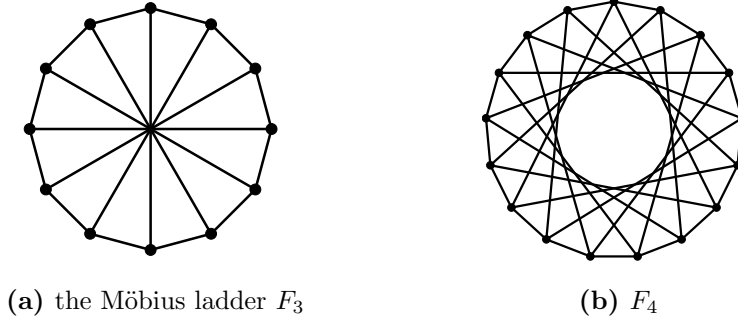


Figure 2: examples of graphs F_k

As our second main result, we determine the structure of $\{C_3, C_5\}$ -free graphs on n vertices with minimum degree larger than $n/5$, thus answering a question of Messuti and Schacht [17].

Theorem 3. *Let G be a $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G is homomorphic to F_k , for some k .*

As a consequence of Theorem 3, we are able to obtain the following.

Theorem 4. *Let G be a $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > \frac{k}{5k-3}n$. Then G is homomorphic to F_{k-1} .*

As F_k is not homomorphic to F_{k-1} , a suitable blow-up of F_k shows that this result is tight.

For $\{C_3, C_5\}$ -free graphs and graphs of higher odd-girth, similar results have been obtained before. Häggkvist and Jin [11] proved that any n -vertex $\{C_3, C_5\}$ -free graph with minimum degree larger than $n/4$ is homomorphic to C_7 ($= F_{2,3}$). Theorem 3 can therefore be seen as the analogue of this result when the minimum degree condition is significantly weakened. Finally, we mention that Messuti and Schacht [17], and Brandt and Ribe-Baumann [4] generalised Häggkvist and Jin's result to $\{C_3, \dots, C_{2\ell-1}\}$ -graphs for $\ell \geq 4$, showing that if the minimum degree is greater than $\frac{3n}{4\ell}$, then such a graph is homomorphic to $C_{2\ell+1}$ ($= F_{2,\ell}$).

1.1 Organisation

The remainder of this paper is organised as follows. In Section 2 we shall provide an outline of the technical results needed to prove our main theorem. Many of these state that certain

subgraphs cannot appear in maximal $\{C_3, C_5\}$ -free graphs of minimum degree larger than $n/5$. In the next three sections (Section 3 to Section 5) we shall prove each of these technical results. In Section 6, we deduce our main theorem, Theorem 3. Finally, Section 7 includes our results concerning homomorphism thresholds, Theorem 1 and Corollary 2.

1.2 Notation

Our notation is standard. In particular, for a graph G , we use $|G|$ to denote the number of vertices of G , $V(G)$ denotes the vertex set, $E(G)$ the edge set, and $\delta(G)$ denotes the minimum degree. For a vertex v , $N_G(v)$ denotes the neighbourhood of v , and for a subset $X \subseteq V(G)$, $N_G(v, X)$ denotes the neighbourhood of v in X , i.e. $N_G(v, X) = N_G(v) \cap X$. We shall often omit the use of the subscript ‘ G ’ when there is no ambiguity. If $X, Y \subseteq V(G)$, then we say an edge e is an $X - Y$ edge if one endpoint of e is in X , the other in Y . If $X = \{x\}$, then we simply say e is an $x - Y$ edge. We denote by $(v_1 \dots v_\ell)$ the cycle on vertices v_1, \dots, v_ℓ taken in this order. Similarly, we denote by $v_0 \dots v_\ell$ the path on vertices v_0, \dots, v_ℓ taken in this order. A cycle (path) with ℓ edges is an ℓ -cycle (ℓ -path).

2 Overview

In this section we shall provide a tour through the technical results needed to establish our main theorem regarding the structure of $\{C_3, C_5\}$ -free graphs of large minimum degree. Note that in proving Theorem 3 we may assume our graph is *maximal* $\{C_3, C_5\}$ -free. Accordingly, the following results concern maximal $\{C_3, C_5\}$ -free graphs. The main tool needed for the proof of Theorem 3 is the following result.

Theorem 5. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then every vertex in G has a neighbour in every 7-cycle in G .*

We remark that Jin [13] proved the analogous theorem for 5-cycles in triangle-free graphs of large enough minimum degree.

In order to establish Theorem 5 we shall need to know that certain subgraphs cannot appear in maximal $\{C_3, C_5\}$ -free graphs of large minimum degree. The first of these results, which shows that $\{C_3, C_5\}$ -free graphs with large minimum degree do not have induced 6-cycles, proves very useful, and we shall use it throughout the paper. Brandt and Ribe-Baumann [4] mention it without proof.

Theorem 6. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G does not contain an induced 6-cycle.*

We shall also need the fact that a ‘partial’ Möbius ladder cannot appear as a subgraph in the graphs we consider. More precisely, we prove the following theorem.

Theorem 7. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G does not contain a 12-cycle with at most five diagonals, two of which are consecutive.*

Of course, by a pair of *consecutive* diagonals in an even cycle $(x_1 \dots x_{2\ell})$ we mean a pair of edges of the form $x_i x_{i+\ell}$, $x_{i+1} x_{i+1+\ell}$ (indices computed modulo 2ℓ). We also remark that there is some ambiguity regarding the statement of Theorem 7. Indeed, suppose that $C = (x_1 \dots x_{12})$ is a 12-cycle with consecutive diagonals $x_1 x_7$ and $x_2 x_8$. Then there is another choice for a 12-cycle, which also has two consecutive diagonals: $C' = (x_2 x_3 \dots x_7 x_1 x_{12} x_{11} \dots x_8)$ with consecutive diagonals $x_1 x_2$ and $x_7 x_8$. Theorem 7 then says that either C or C' must have all of its diagonals present. This ambiguity will not be a hindrance during the proofs of our results.

We note, and prove, the following useful corollary of Theorem 7.

Corollary 8. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. If u is a vertex with no neighbours in a 7-cycle C , then u has no neighbour with two neighbours in C .*

Proof. Suppose that $C = (x_1 \dots x_7)$ and u has no neighbours in C , but a neighbour v of u has two neighbours in C . Without loss of generality, we may assume that v is adjacent to x_2 and x_7 (see Figure 3a).

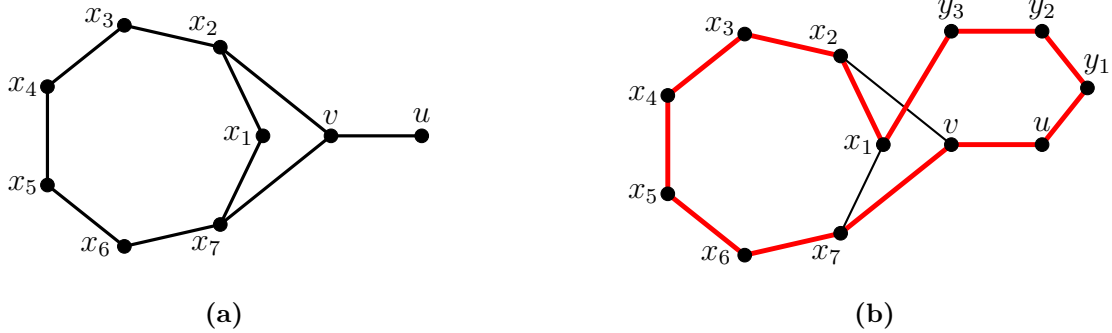


Figure 3: u has no neighbours in C ; its neighbour v has two neighbours in C

Since u is not adjacent to x_1 , there must be a path of length 2 or 4 between them, but 2 is impossible (it will complete the path uvx_2x_1 to a 5-cycle), so there is a 4-path $uy_1y_2y_3x_1$. One may check that none of y_1, y_2, y_3 is equal to one of the vertices of C or to u or v (see Figure 3b). But then $(x_1 \dots x_7 v u y_1 y_2 y_3)$ is a 12-cycle with two consecutive diagonals $x_1 x_7$ and $x_2 v$ (see the red cycle in Figure 3b). It follows from Theorem 7 that all diagonals in the cycle must be present (or we need to consider the 12-cycle $(x_2 \dots x_7 x_1 y_3 y_2 y_1 u v)$ with diagonals $x_1 x_2$ and $x_7 v$). In particular, u has a neighbour in C , a contradiction. \square

Finally, in order to prove Theorem 5, we establish the following, which is the last of our results regarding forbidden subgraphs in maximal $\{C_3, C_5\}$ -free graphs of large minimum degree.

Theorem 9. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G does not contain, as an induced graph, the graph obtained by two 7-cycles whose intersection is a path of length 3.*

Before proceeding to the proofs of the above results, we shall show how to prove Theorem 5 using Theorem 6, Corollary 8, and Theorem 9. Before doing this, let us make the following observation:

Observation 10. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Suppose that u has no neighbours in a 7-cycle C . Then u has a common neighbour with at most one of the vertices in C .*

Proof. Suppose that u has no neighbour in the cycle $C = (x_1 \dots x_7)$. Furthermore, suppose that u has a common neighbour v with x_1 (see Figure 4). By symmetry, it suffices to show that u has no

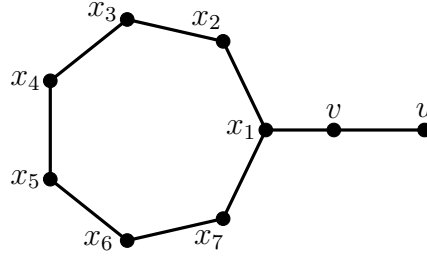


Figure 4: u and x_1 have a common neighbour v

common neighbours with x_2, x_3 or x_4 . It easily follows that u and x_2 have no common neighbours (otherwise, a cycle of length 3 or 5 is formed). Suppose that u and x_3 have a common neighbour w . Consider the 6-cycle $(vuwx_3x_2x_1)$. Recall that G has no induced 6-cycles, thus one of the pairs ux_2, vx_3, wx_1 is an edge in G . But ux_2 is not an edge, by the assumption that u has no neighbour in C , and if one of vx_3 and wx_1 is an edge, a contradiction to Corollary 8 is reached. Finally, if u and x_4 have a common neighbour w then the set $\{u, v, w, x_1, \dots, x_7\}$ induces a graph that consists of two 7-cycles whose intersection is a path of length 3, contradicting Theorem 9. \square

We are now ready to prove Theorem 5.

Theorem 5. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then every vertex in G has a neighbour in every 7-cycle in G .*

Proof. Suppose that the theorem is false and choose a vertex u and a 7-cycle C which minimise the distance between u and C such that u has no neighbour in C . It easily follows that there is a path of length two between u and C . Indeed, if P is a $u - C$ path of minimum length, then u 's neighbour on P must be adjacent to a vertex of C : otherwise, the minimality in the choice of u and

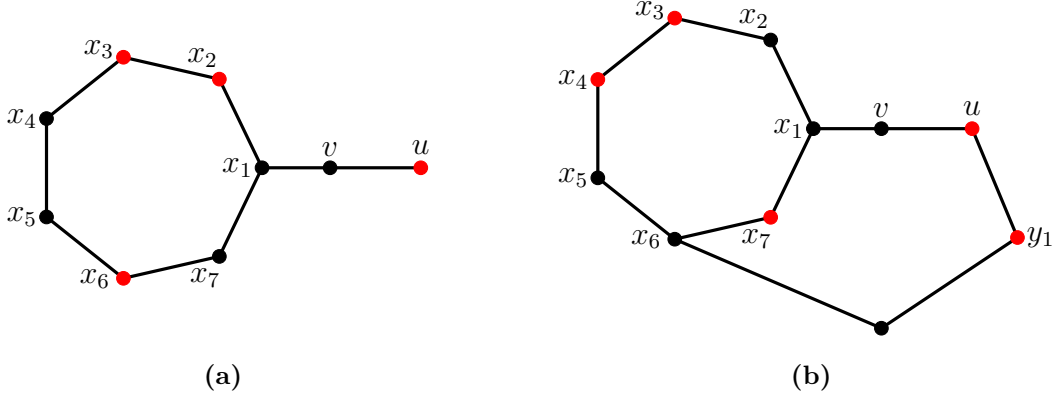


Figure 5: proof of Theorem 3

C is contradicted. Therefore, we may assume without loss of generality that u has no neighbours in the 7-cycle $C = (x_1 \dots x_7)$ and v is a common neighbour of u and x_1 (see Figure 5a).

Since u is not joined to x_2 , there is a 4-path $uy_1y_2y_3x_2$ between u and x_2 . We note that y_1 cannot be joined to x_1 (otherwise, a 5-cycle is formed). Thus, by Observation 10, y_1 has no neighbours in C . We note that no two of the four vertices $\{u, x_2, x_3, x_6\}$ have a common neighbour (see the red vertices in Figure 5a; this follows from Observation 10 and the assumption that G is $\{C_3, C_5\}$ -free). It follows, from the minimum degree condition, that y_1 has a common neighbour with one of u, x_2, x_3, x_6 . But y_1 does not have a common neighbour with either u or x_2 (otherwise, a cycle of length 3 or 5 is formed). Thus y_1 has a common neighbour with either x_3 or x_6 . Without loss of generality, we assume that y_1 has a common neighbour with x_6 (see Figure 5b). Then, by Observation 10, y_1 has no common neighbours with any other vertex in C . It follows that no two of the vertices $\{u, y_1, x_3, x_4, x_7\}$ have a common neighbour (see the red vertices in Figure 5b), a contradiction to the minimum degree condition. \square

In the next three sections we shall prove all aforementioned results concerning forbidden subgraphs (Theorems 6, 7 and 9). The general strategy of such proofs is the following. We want to show that some graph F cannot appear in a maximal $\{C_3, C_5\}$ -free graph G of large minimum degree. If F is a subgraph of G , and if every vertex has a ‘small’ number of neighbours in F , then double counting the edges between $V(F)$ and $V(G) \setminus V(F)$ will produce a contradiction with the minimum degree condition. However, much of the time the original target graph F will not satisfy this goal, and we shall need to pass to some suitable subgraph of F which meets our needs. This often requires detailed analysis of the possible neighbourhoods of vertices in F (or some subgraph of F). In order to aid in this investigation, we introduce the following definition.

Definition 11. A subgraph H of a graph G is called *well-behaved* (in G) if for every vertex u in G , there is a vertex v in H , such that the neighbourhood $N(u, V(H))$ of u in H is contained in the neighbourhood of v .

Many of the subgraphs we consider are actually well-behaved (in their respective host graphs).

3 No induced 6-cycles

It turns out that maximal $\{C_3, C_5\}$ -free graphs of high minimum degree forbid induced 6-cycles (see [4] for a statement without proof), and we shall use this extensively in the remainder of the paper. Our aim in this section is to establish this fact.

Theorem 6. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G does not contain an induced 6-cycle.*

Proof. Suppose otherwise, that G contains an induced 6-cycle $C = (x_1 \dots x_6)$. Owing to the missing edge x_1x_4 , and by the edge-maximality of G , there must be a 4-path $P_{14} = x_1y_1y_2y_3x_4$ (a 2-path is impossible, as it would create a C_5). It is easy to verify that $y_1, y_2, y_3 \notin V(C)$. Similarly, the missing edge x_2x_5 implies the existence of a 4-path $P_{25} = x_2z_1z_2z_3x_5$ in G with $z_1, z_2, z_3 \notin V(C) \cup V(P_{14})$, and the missing edge x_3x_6 implies the existence of a 4-path $P_{36} = x_3w_1w_2w_3x_6$ with $w_1, w_2, w_3 \notin V(C) \cup V(P_{14}) \cup V(P_{25})$. Denote by H the graph induced on $V(C) \cup V(P_{14}) \cup V(P_{25}) \cup V(P_{36})$ (see Figure 6). We shall show that G cannot contain H as a subgraph. The proof breaks into two cases:

1. At least two vertices from $\{y_2, z_2, w_2\}$ have a common neighbour.
2. No pairs from $\{y_2, z_2, w_2\}$ have a common neighbour.

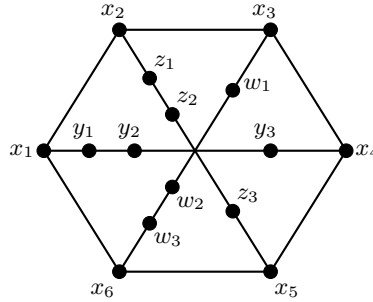


Figure 6: H

In each case, we shall find a 10-vertex subgraph of H for which every vertex of G has at most two neighbours in H . We reach a contradiction to the minimum degree condition on G via double counting the edges between this subgraph and the rest of G .

3.1 Case 1

Suppose, without loss of generality, that y_2 and z_2 have a common neighbour v . Denote by H' the 10-vertex graph induced on $V(H) \setminus V(P_{36})$ (see Figure 7).

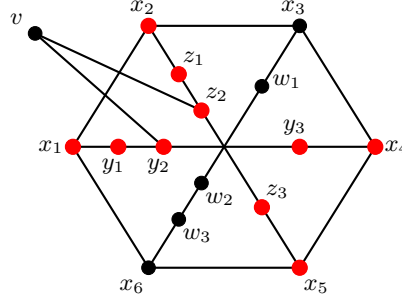


Figure 7: H' , with common neighbour v

Claim 12. *Every vertex of G has at most 2 neighbours in $V(H')$.*

Proof. We shall examine possible neighbourhoods in H' of a vertex according to the number of neighbours it has in $V(C) \setminus \{x_3, x_6\}$. Observe that x_1 and x_4 cannot have a common neighbour—if otherwise, and u is a common neighbour, then the 5-cycle $(x_1 u x_4 x_5 x_6)$ is formed. Similarly, x_2 and x_5 do not have a common neighbour. Thus, a vertex in G can have at most two neighbours in $V(C) \setminus \{x_3, x_6\}$, and if it has precisely two such neighbours, then it is either joined to both x_2 and x_4 or to both x_1 and x_5 . By symmetry, suppose the former is true, and let u be a vertex joined to x_2 and x_4 . It is then routine to check that u cannot be joined to any other vertex of H' : all cases lead to the formation of a triangle or pentagon in G .

Let us now consider vertices which have precisely one neighbour in $V(C) \setminus \{x_3, x_6\}$. By symmetry, let u be a vertex joined to x_1 . We claim that u can be joined to at most one vertex, which must be from $\{y_2, z_1\}$. Indeed, it is easy to verify that u cannot be joined to any vertices of $H' \setminus \{y_2, z_1\}$, since these cases lead to the formation of a triangle or pentagon in G . Suppose u is adjacent to both y_2 and z_1 . This, however, produces the 5-cycle $(y_2 u z_1 z_2 v)$, a contradiction.

Finally, we consider vertices which have no neighbours in $V(C) \setminus \{x_3, x_6\}$. First, note that if a vertex u is joined to y_2 , then its only other possible neighbour in H' is z_2 (and the same claim holds with the roles of y_2 and z_2 reversed). For example, if u is joined to y_2 and z_3 , the 5-cycle $(y_2 u z_3 z_2 v)$ is produced. One may dispense with the other cases similarly. On the other hand, both pairs y_3, z_3 and y_1, z_1 do not have any common neighbours. It follows that any vertex with no neighbours in $V(C) \setminus \{x_3, x_6\}$ has at most two neighbours in H' , and this finishes the proof of Claim 12. \square

Let us estimate the number of edges between $V(H')$ and $V(G) \setminus V(H')$ in two ways. Using the minimum degree condition, there are more than $10(n/5 - 2) = 2(n - 10)$ such edges (indeed, Claim 12 implies, in particular, that every vertex in H' has at most two neighbours in H' , so it has more than $n/5 - 2$ neighbours outside of H'). But Claim 12 implies that there are at most $2(n - 10)$ such edges, a contradiction. This completes the proof of Theorem 6 under Case 1.

3.2 Case 2

Suppose that no pairs from $\{y_2, z_2, w_2\}$ have a common neighbour. We begin by examining the size and structure of possible neighbourhoods in H of vertices of G .

Let u be a vertex which is not joined to any vertex in $V(C)$. If u is joined to a middle vertex, say y_2 , then by assumption it cannot be adjacent to z_2 or w_2 . Further, u cannot be joined to y_1 or y_3 (else, a triangle is formed), has at most one neighbour in $\{z_3, w_3\}$, and has at most one neighbour in $\{z_1, w_1\}$. Thus, u has at most 3 neighbours in H . Similarly, one may verify that if u has no neighbour in $\{y_2, z_2, w_2\}$, then u has at most 3 neighbours in H as well.

Suppose u is a vertex joined to 2 vertices of C . Say, by symmetry, that u is joined to x_2 and x_4 . Then it is easy to check that the only other possible neighbour of u in H is w_1 . Hence u has at most 3 neighbours in H .

If u is a vertex joined to 3 neighbours of C , then (up to relabelling) u is adjacent to all vertices in $\{x_1, x_3, x_5\}$, and one may verify that u can have no further neighbours in H . Thus, such vertices have at most 3 neighbours in H .

Only one case remains: suppose u has precisely one neighbour in $V(C)$, and, by symmetry, suppose this neighbour is x_1 . In this case, u may be joined to all vertices in $\{y_2, z_1, w_3\}$. Accordingly, u has at most 4 neighbours in H .

Now, if every vertex of G has at most 3 neighbours in H , then we are done by double counting the edges between $V(H)$ and $V(G) \setminus V(H)$: there are at most $3(n-15)$ such edges, and by the minimum degree condition, more than $15(n/5-3) = 3(n-15)$ such edges, a contradiction. Therefore, we may assume that there is a vertex v of degree 4 in $V(H)$. By the preceding analysis, we may assume that y_1z_1 and y_1w_3 are edges: if not, replace y_1 by v .

The proof breaks into two cases from here:

- (a) z_3 and w_1 have a common neighbour.
- (b) z_3 and w_1 do not have a common neighbour.

Assuming (a), let w be a common neighbour of z_3 and w_1 , and denote by H'' the 10-vertex graph induced on $V(H) \setminus V(P_{14})$ (see Figure 8).

Claim 13. *Every vertex of G has at most 2 neighbours in H'' .*

Proof. The proof is similar to that of Claim 12. Any vertex u is adjacent to at most 2 vertices of $V(C) \cap V(H'')$. If u is joined to x_2 and x_6 , then its only other possible neighbour is y_1 , but $y_1 \notin V(H'')$. A similar statement holds if u is joined to x_3 and x_5 .

Suppose now that u has precisely one neighbour in $V(C) \cap V(H'')$, and suppose this neighbour is x_2 . By our preceding analysis of possible neighbourhoods in H , u 's only other possible neighbours

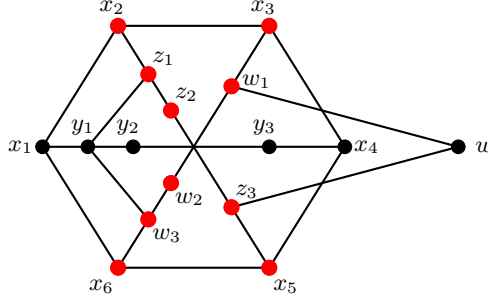


Figure 8: H'' , with common neighbour w

are y_1, z_2 , and w_1 . However, $y_1 \notin V(H'')$ and u cannot be joined to both z_2 and w_1 : otherwise, the 5-cycle $(z_2 z_3 w w_1 u)$ is formed. Hence, u is joined to at most 2 vertices of H'' .

Similarly, if u is a vertex whose only neighbour in $V(C) \cap V(H'')$ is x_3 , then u 's only other possible neighbours are y_3, w_2 , and z_1 . But $y_3 \notin V(H'')$ and u cannot be joined to both w_2 and z_1 : otherwise the 5-cycle $(w_2 w_3 y_1 z_1 u)$ is created. The other cases (i.e., u joined to x_5 or x_6) are symmetric.

Finally, suppose u is a vertex with no neighbour in $V(C) \cap V(H'')$. If u is joined to a middle vertex, say, without loss of generality, w_2 , then by assumption u cannot be joined to z_2 . Hence the only other possible neighbours of u are z_1 and z_3 . But u cannot be joined to z_1 , since otherwise $(u w_2 w_3 y_1 z_1)$ is a 5-cycle in G . If u is not joined to a middle vertex, then observe that it can be adjacent to at most one vertex from each pair z_3, w_3 and z_1, w_1 .

This completes the proof of Claim 13. □

Let us now assume (b), that z_3 and w_1 do not have a common neighbour. Denote by H''' the 10-vertex graph induced on $V(H) \setminus \{x_1, x_2, x_6, y_2, y_3\}$ (see Figure 9).

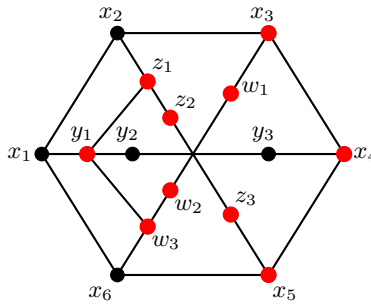


Figure 9: H'''

Claim 14. *Every vertex of G has at most 2 neighbours in H''' .*

Proof. If a vertex u of G has 2 neighbours in $V(C) \cap V(H''')$, then u must be joined to x_3 and x_5 . But u 's only other potential neighbour is y_3 , and $y_3 \notin V(H''')$.

Suppose u is a vertex with exactly one neighbour in $V(C) \cap V(H''')$. First, suppose u is joined to x_3 . Then u 's only other possible neighbours are w_2, y_3 , and z_1 . But $y_3 \notin V(H''')$ and u cannot be joined to both z_1 and w_2 , as otherwise the 5-cycle $(uw_2w_3y_1z_1)$ is in G . The case when u is joined to x_5 is dealt with symmetrically.

Suppose now that u is joined to x_4 . The only other possible neighbours are then y_2, z_3 , and w_1 . Observe that $y_2 \notin V(H''')$, and, by assumption, z_3 and w_1 have no common neighbour, so u is joined to at most one of them.

One may (as in the proof of Claim 13) dispense with the case when u is a vertex with no neighbours in $V(C) \cap V(H''')$. Thus, no vertex of G has more than 2 neighbours in $V(H''')$ and this completes the proof of Claim 14. \square

We may now complete the proof of Theorem 6 in Case 2. Indeed, if (a) holds, then apply Claim 13 together with the usual double counting technique to produce a contradiction. If instead (b) holds, then apply Claim 14 together with double counting. This completes the proof of Theorem 6. \square

4 12-cycles with few diagonals

Our aim in this section is to prove Theorem 7. We divide the proof into steps, according to the number of diagonals. Let us restate the theorem here for convenience.

Theorem 7. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G does not contain a 12-cycle with at most five diagonals, two of which are consecutive.*

Note that the case of five diagonals is immediate from Theorem 6 that forbids induced 6-cycles, and so we have:

Corollary 15. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G has no induced 12-cycle with exactly five diagonals.*

The subsequent subsections deal with the remaining cases.

4.1 No 12-cycle with exactly four diagonals

Lemma 16. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G has no 12-cycle with exactly four diagonals.*

Proof. Suppose that $(x_1 \dots x_{12})$ is a 12-cycle with exactly four diagonals. Let H be the graph induced by $\{x_1, \dots, x_{12}\}$. In light of Theorem 6, G has no induced 6-cycle, so we may assume that the edges $x_1x_7, x_2x_8, x_3x_9, x_4x_{10}$ are present in the graph and that x_5x_{11}, x_6x_{12} are non-edges (see

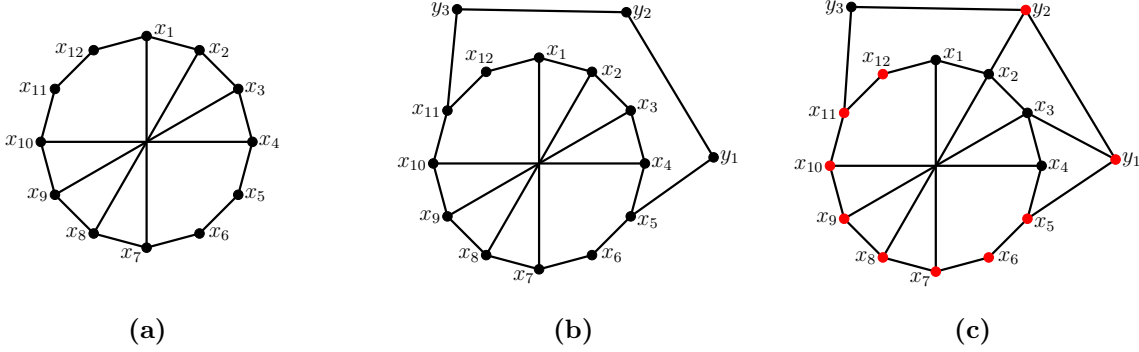


Figure 10: four chords

Figure 10a). In fact, it is easy to verify that the only edges in H are the edges of the 12-cycle and the four aforementioned diagonals.

Recall that a subgraph F of a graph G is called well-behaved if for every vertex u in G there is a vertex v in F such that $N(u, V(F)) \subseteq N(v)$ (see Definition 11). Before returning to the proof, we make the following observation.

Claim 17. *H is well-behaved as a subgraph of G .*

Proof. We first point out that no vertex u in G can be adjacent to all of $\{x_4, x_6, x_{11}\}$. Indeed, otherwise, $(ux_{11}x_{12}x_1x_7x_6)$ is an induced C_6 (the addition of any chord to this cycle creates a triangle or a pentagon), contradicting Theorem 6. By symmetry, no vertex can be adjacent to all vertices in one of the following sets: $\{x_5, x_7, x_{12}\}$, $\{x_1, x_6, x_{11}\}$, $\{x_5, x_{10}, x_{12}\}$. We conclude that no vertex can be adjacent to both x_6 and x_{11} . Indeed, by considering the 6-cycle $(x_1x_7x_6ux_{11}x_{12})$, since there is no induced C_6 , u must be adjacent to x_1 , contradicting the above. Similarly, no vertex is adjacent to both x_5 and x_{12} . One may check that any other possible neighbourhood of a vertex of G in H is contained in the neighbourhood of a vertex in H . \square

The pair x_5x_{11} is a non-edge in G , and so, by the assumption that G is maximal $\{C_3, C_5\}$ -free, there is a path of length two or four between x_5 and x_{11} . In fact, the length must be four because, otherwise, a cycle of length 3 or 5 will be created. Let $x_5y_1y_2y_3x_{11}$ be this 4-path (see Figure 10b). One may verify that $y_1, y_2, y_3 \notin V(H)$.

Claim 18. *We may assume that y_1x_3 is an edge in G .*

Proof. No two of the following vertices have a common neighbour: x_3, x_6, x_9, x_{12} (they are at distance one or three from each other). In other words, their neighbourhoods are pairwise disjoint, and so, by the minimum degree condition, every vertex in G has a common neighbour with at least one of these four vertices. Note that y_2 does not have a common neighbour with either x_6 or x_{12} (this will create a C_5). By symmetry, we may assume that y_2 and x_3 have a common neighbour

u. If $u = y_1$, Claim 18 follows. Thus, we suppose otherwise. Consider the 6-cycle $(uy_2y_1x_5x_4x_3)$. Since there are no induced 6-cycles, one of the following is an edge: y_1x_3, y_2x_4, ux_5 . If y_1x_3 is an edge, the claim follows; y_2x_4 cannot be an edge (because of the 5-cycle $(y_2x_4x_{10}x_{11}y_3)$); if ux_5 is an edge, we replace y_1 by u to obtain the required property. \square

Claim 19. *We may assume that y_2x_2 is an edge.*

Proof. As before, by considering the neighbours of x_2, x_5, x_8, x_{11} , we have that y_3 has a common neighbour with x_2 or x_8 . If u is a common neighbour of y_3 and x_2 , we may assume that $u \neq y_2$ (otherwise, we are done). By considering the 6-cycle $(ux_2x_3y_1y_2y_3)$, either y_2x_2 or uy_1 is an edge. We may assume that uy_1 is an edge. Then, by replacing y_2 by u we obtain the required property. Now suppose that y_3 and x_8 have a common neighbour u . By considering $(ux_8x_9x_{10}x_{11}y_3)$, u is adjacent to x_{10} . This, in turn, implies that u is adjacent to x_3 (see $(ux_8x_2x_3x_4x_{10})$), a contradiction: the 5-cycle $(ux_3y_1y_2y_3)$ is formed. \square

Denote by H' the graph induced by $\{x_5, \dots, x_{12}, y_1, y_2\}$ (see the red vertices in Figure 10c). We shall show that every vertex of G has few neighbours in H' , yielding a contradiction to the minimum degree condition on G . More precisely, we have the following:

Claim 20. *No vertex in G has more than two neighbours in H' .*

Proof. We note that by Claim 17, no vertex in G has more than two neighbours in $V(H') \cap V(H)$. Thus, if a vertex u has three neighbours in H' , at least one of them is either y_1 or y_2 . If u is adjacent to y_1 , then the only other neighbours u can have in H' are x_6, x_9, x_{12} , but no two of these vertices may have a common neighbour. Similarly, if u is adjacent to y_2 , its other possible neighbours in H' are x_5, x_8, x_{11} , no two of which have a common neighbour. The claim follows. \square

Using Claim 20, we may now finish the proof of Lemma 16 by double counting the number of edges between H' and $V(G) \setminus V(H')$, as usual. \square

4.2 No 12-cycle with two or three diagonals

In this subsection we deal with the remaining case, of a 12 cycle with two or three diagonals, and thereby complete the proof of Theorem 7.

Lemma 21. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G induces no 12-cycle with two diagonals and at most one additional chord.*

Proof. Suppose that $C = (x_1 \dots x_{12})$ is a 12-cycle with two consecutive diagonals x_1x_7 and x_2x_8 , and at most one additional chord. We note that any additional chord is a diagonal in one of the following 12-cycles $(x_1 \dots x_{12})$ or $(x_2 \dots x_7x_1x_{12} \dots x_8)$, both of which have two consecutive

diagonals. Hence, and by symmetry, we may assume that the additional chord is either x_6x_{12} or x_5x_{11} . However, if x_5x_{11} is the additional chord, then $(x_1x_7x_6x_5x_{11}x_{12})$ is an induced 6-cycle, contradicting Theorem 6. Thus we assume that, if there is an additional chord, it is x_6x_{12} (see Figure 11). Furthermore, if x_6x_{12} is not an edge, we assume that G contains no 12-cycles with two consecutive diagonals and exactly one extra chord.

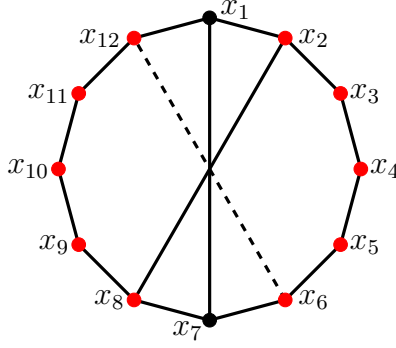


Figure 11: two or three chords

Let H be the graph induced by $\{x_1, \dots, x_{12}\}$ and denote $H' = H \setminus \{x_1, x_7\}$ (see the red vertices in Figure 11).

Claim 22. *No vertex in G has at least three neighbours in H' .*

Proof. Observe that for any 7-cycle C and any vertex u in G , u has at most two neighbours in C , and if it does have two neighbours, they are at distance 2 in C . Suppose that u has three neighbours in H' . It follows by symmetry that u has two neighbours in $\{x_2, \dots, x_6\}$, which we can denote by x_{i-1} and x_{i+1} for some $i \in \{3, 4, 5\}$, and another neighbour x_j for some $j \in \{8, \dots, 12\}$. But then, by replacing x_i by u , we may assume that x_i is joined to x_j . This is a contradiction: either to Lemma 16 (if C had three chords, i.e. if x_6x_{12} is an edge, then now it has four chords); or, if x_6x_{12} is not an edge, to the assumption that there is no 12-cycle with two consecutive diagonals and an additional chord. \square

The proof of Lemma 21 follows from Claim 22 by double counting the number of edges between H' and $V(G) \setminus V(H')$. \square

5 Two 7-cycles intersecting in a 3-path

In this section we prove Theorem 9; that is, the graph in Figure 12 cannot appear as an induced subgraph of a maximal $\{C_3, C_5\}$ -free graph on n vertices and minimum degree larger than $n/5$. The proof follows the same strategy as the proofs in the previous two sections, though it requires more effort.

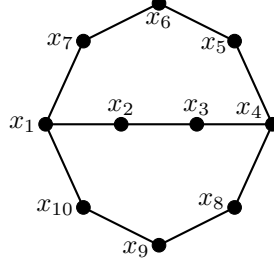


Figure 12: two 7-cycles intersecting in a 3-path

Theorem 9. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. Then G does not contain, as an induced graph, the graph obtained by two 7-cycles whose intersection is a path of length 3.*

Proof. Suppose that H is an induced subgraph of G which is the union of two 7-cycles intersecting in a path of length 3. Denote the two 7-cycles by $(x_1x_2x_3x_4x_5x_6x_7)$ and $(x_1x_2x_3x_4x_8x_9x_{10})$ (see Figure 12). We start by showing that H is a well-behaved subgraph of G (recall Definition 11), a fact that will be useful in the proof.

Claim 23. *The graph H is well-behaved.*

Proof. Suppose that H is not well-behaved. Then, up to relabelling, one of the two following pairs has a common neighbour in G : $\{x_6, x_9\}$ or $\{x_5, x_{10}\}$. If u is a neighbour of x_6 and x_9 then, by Theorem 6, u is also a neighbour of x_1 (consider the 6-cycle $(ux_6x_7x_1x_{10}x_9)$). But then, x_9 has no neighbours in the 7-cycle $C = (x_1 \dots x_7)$ and its neighbour u has two neighbours in C (x_1 and x_6), contradicting Corollary 8. Now suppose that u is a neighbour of both x_5 and x_{10} . Consider the 6-cycle $(ux_5x_4x_8x_9x_{10})$. By Theorem 6, G has no induced C_6 , so u must be adjacent to x_8 . Now consider the 7-cycle $(ux_{10}x_1x_2x_3x_4x_8)$. The vertex x_6 has no neighbours in C (x_6 cannot be adjacent to u), but x_5 has two neighbours in C (x_4 and u). This is a contradiction to Corollary 8. \square

Arguments as in Claim 23, using Corollary 8 and Theorem 6 will appear frequently in the proof of Theorem 9.

Since x_6 and x_8 are nonadjacent, there is a 4-path with ends x_6 and x_8 (a 2-path would create a C_5). Up to relabelling, three cases arise:

1. There is a 3-path $x_6y_1y_2x_9$ between x_6 and x_9 . The vertices y_1 and y_2 are not in H .
2. There is a 3-path $x_7y_1y_2x_8$ between x_7 and x_8 . The vertices y_1 and y_2 are not in H .
3. There is a 4-path $x_6y_1y_2y_3x_8$ between x_6 and x_8 . The vertices y_1, y_2, y_3 are not in H .

In the rest of the proof, we show that each of the three cases is impossible, thus completing the proof of Theorem 9. Case 2 will be the most difficult to resolve.

5.1 Case 1: a 3-path between x_6 and x_9

Denote by H' the graph induced by $\{x_1, \dots, x_{10}, y_1, y_2\}$ (see Figure 13).

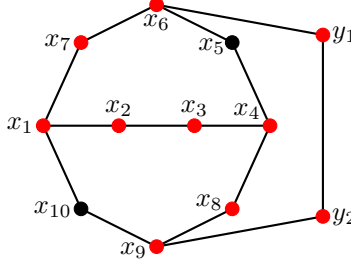


Figure 13: Case 1: the graphs H' and H''

Claim 24. *The graph H' is well-behaved.*

Proof. Suppose that H' is not well-behaved. Up to relabelling, it follows that y_1 and x_3 have a common neighbour u (recall that H is well-behaved). By considering the 6-cycle $(ux_3x_4x_5x_6y_1)$ and in light of Theorem 6, it follows that u is adjacent to x_5 . Consider the 7-cycle $C = (x_1x_2x_3ux_5x_6x_7)$. Observe that x_4 has two neighbours in C (x_3 and x_5), but x_8 has no neighbours in C (x_8 cannot be adjacent to u). This contradicts Corollary 8. \square

Now consider the graph $H'' = H' \setminus \{x_5, x_{10}\}$ (see the red vertices in Figure 13). It follows from Claim 24 that every vertex in G has at most two neighbours in H'' . The usual argument, of double counting the edges between H'' and $V(G) \setminus V(H'')$, leads to a contradiction to the minimum degree condition, thus completing the proof of Theorem 9 in Case 1.

5.2 Case 2: a 3-path between x_7 and x_8

Denote by H' the graph induced by $\{x_1, \dots, x_{10}, y_1, y_2\}$ (see Figure 14).

Claim 25. *The graph H' is well-behaved.*

Proof. If H' is not well-behaved, then up to relabelling, y_1 and x_3 have a common neighbour u (recall that H is well-behaved by Claim 23). Consider the 6-cycle $(uy_1x_7x_1x_2x_3)$. Since there is no induced 6-cycle (Theorem 6), either y_1 is adjacent to x_2 , or u is adjacent to x_1 . The former case leads to a contradiction similarly to Claim 24: then x_1 has two neighbours in the 7-cycle $(x_2x_3x_4x_5x_6x_7y_1)$ whereas its neighbour x_{10} has no neighbours there, contradicting Corollary 8. So, suppose the latter case holds, i.e. u is adjacent to x_1 . But then u has two neighbours in the 7-cycle $(x_1x_2x_3x_4x_8x_9x_{10})$ whereas y_1 has none, a contradiction. \square

As before, in light of the missing edge x_6x_{10} , one of the following three cases holds.

induced by $\{x_1, \dots, x_{10}, z_1, z_2\}$; see Claim 23). By symmetry, we may assume that u is adjacent to y_1 . Suppose that u is also adjacent to z_1 . By considering the 6-cycle $(uz_1x_5x_6x_7y_1)$, it follows that u is adjacent to x_6 . But, then, x_7 has two neighbours in the 7-cycle $(uy_1y_2x_8x_4x_5x_6)$ while x_1 has none. This is a contradiction to Corollary 8.

It remains to consider the case where u is adjacent to both y_1 and z_2 . It follows that u is adjacent to x_1 (consider $(uy_1x_7x_1x_{10}z_2)$). This is a contradiction to Corollary 8: x_{10} has two neighbours in $(uz_2z_1x_5x_6x_7x_1)$ whereas x_9 has none. \square

Claim 26 leads to a contradiction by double counting the edges between F' and $V(G) \setminus V(F')$. This completes the proof of Theorem 9 in this case.

Case 2c: a 3-path between x_7 and x_8 and a 4-path between x_6 and x_{10}

Denote by F the graph induced by $\{x_1, \dots, x_{10}, y_1, y_2, z_1, z_2, z_3\}$ (see Figure 16).

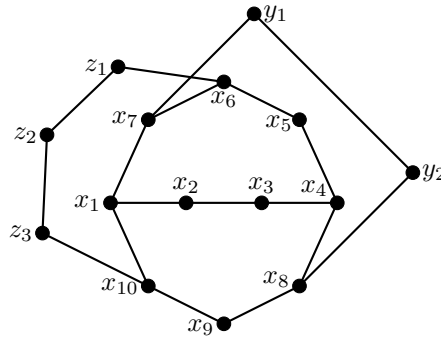


Figure 16: Case 2c: the graph F

Claim 27. *The only edges spanned by F are those spanned by H and the edges of the two paths $x_6z_1z_2z_3x_{10}$ and $x_7y_1y_2x_8$.*

Proof. First we note that y_1 and y_2 do not have additional neighbours in $\{x_1, \dots, x_{10}\}$. Indeed, by symmetry we assume that y_1 has an additional neighbour in H . The only possible such neighbour is x_2 . We reach a contradiction to Corollary 8 (consider the 7-cycle $(y_1x_2x_3x_4x_5x_6x_7)$ and the vertices x_1 and x_{10}).

We now show that z_1 , z_2 and z_3 do not have additional edges into H . Using the fact that H' is well-behaved, the only possible additional neighbour of z_1 is x_4 . But then, by replacing x_5 by z_1 , we may assume that there is a 3-path from x_5 to x_{10} . This leads to a contradiction, as we have seen in Case 2b. Similarly, the possible additional neighbours of z_3 in H are x_8 and x_2 . If z_3 is adjacent to x_8 then, by replacing x_9 by z_3 , we may assume that there is a 3-path between x_6 and x_9 , contradicting Case 1. If z_3 is adjacent to x_2 we reach a contradiction to Corollary 8 (x_1 has

two neighbours in the 7-cycle $(z_3x_2x_3x_4x_8x_9x_{10})$ while x_7 has none). The possible neighbours of z_2 in H are x_3 , x_5 and x_9 . But z_2 is not adjacent to x_5 or x_9 , because, otherwise, there is a 3-path between x_5 and x_{10} or between x_6 and x_9 , contradicting previous cases. Furthermore, z_2 is not adjacent to x_3 because, otherwise, $(z_1z_2x_3x_4x_5x_6)$ is an induced 6-cycle, contradicting Theorem 6.

Finally, we show that there are no edges between $\{z_1, z_2, z_3\}$ and $\{y_1, y_2\}$. The only such edges that do not create a triangle or pentagon are z_1y_1 and z_2y_2 . If z_1 is adjacent to y_1 we reach a contradiction to Corollary 8 (see $(z_1y_1y_2x_8x_4x_5x_6)$ and the vertices x_1, x_7), and if z_2y_2 is an edge, a contradiction to Theorem 6 is reached (consider the induced 6-cycle $(z_1z_2y_2y_1x_7x_6)$). This completes the proof of Claim 27. \square

The following claim states that no vertex has more than three neighbours in F . Since $|F| = 15$, this is a contradiction to the minimum degree condition on G by the usual double counting argument, hence the proof of Theorem 9 in this case follows.

Claim 28. *No vertex has more than three neighbours in F .*

Proof. Since H' is well-behaved (see Claim 25; recall that H' is the graph induced by the set $\{x_1, \dots, x_{10}, y_1, y_2\}$) and has maximum degree 3, if there is a vertex u with four neighbours in F , it must be adjacent to at least one of z_1, z_2, z_3 . We note that u cannot be adjacent to both z_1 and z_3 because then, by replacing z_2 by u , we may assume that z_2 has an additional edge in F , a contradiction to Claim 27. It follows that u has one neighbour among z_1, z_2, z_3 and at least three neighbours in H' . Since H' is well-behaved, u is adjacent to all three neighbours of a vertex v in H' of degree three (in H'). But then, by replacing v by u , we may assume that v has an additional edge in F , a contradiction to Claim 27. \square

5.3 Case 3: a 4-path between x_6 and x_8

Denote by H' the graph induced by $\{x_1, \dots, x_{10}, y_1, y_2, y_3\}$, and let $H'' = H' \setminus \{x_5, x_7, y_3\}$ (see Figure 17).

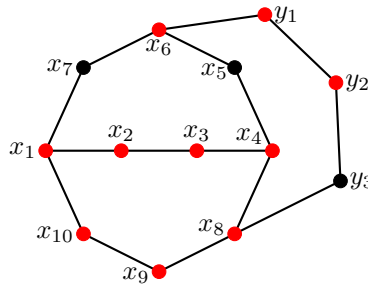


Figure 17: Case 3: the graphs H' and H''

Claim 29. *The only edges in H' are those spanned by H or by the path $x_6y_1y_2y_3x_8$.*

Proof. Suppose that there are additional edges. These must be between $\{y_1, y_2, y_3\}$ and $V(H)$. The only possible neighbour (that is not already accounted for) of y_1 in H' is x_1 . But then, by replacing x_7 by y_1 , we reach a contradiction to Case 2.

The only possible additional neighbours of y_3 in H are x_3 and x_{10} . If y_3 is adjacent to x_3 , then x_4 has two neighbours in $(x_1x_2x_3y_3x_8x_9x_{10})$ whereas x_5 has none, a contradiction to Corollary 8. If y_3 is adjacent to x_{10} then, by replacing x_9 with y_3 , we reduce to Case 1.

The only possible additional neighbours of y_2 in H are x_2, x_7, x_9 . If y_2 is adjacent to x_7 or x_9 we reduce to previous cases. Finally, if y_2 is adjacent to x_2 then $(x_6x_7x_1x_2y_2y_1)$ is an induced 6-cycle, a contradiction to Theorem 6. \square

Claim 30. *No vertex in G has more than two neighbours in H'' .*

Proof. Suppose that there is a vertex u in G with three neighbours in H'' . Since H is well-behaved (see Claim 23), u must be a neighbour of either y_1 or y_2 .

Suppose first that u is a neighbour of y_1 . The other possible neighbours of u in H'' are x_2, x_3, x_9, x_{10} . Out of these four vertices, the only two that may have a common neighbour are x_2 and x_{10} . By considering the 6-cycle $(ux_2x_1x_7x_6y_1)$, it follows that u is adjacent also to x_7 , i.e. u is adjacent to x_2, x_7, x_{10}, y_1 . By replacing x_1 by u , we may assume that y_1 is adjacent to x_1 , a contradiction to Claim 29.

We may now assume that u is adjacent to y_2 . The other possible neighbours of u in H'' are $x_1, x_2, x_3, x_6, x_8, x_{10}$. If u is adjacent to x_6 or x_8 , then by replacing y_1 or y_3 by u we see that u cannot have any additional neighbours in H'' : otherwise we reach a contradiction to Claim 29. It follows that u is not adjacent to x_1 , because otherwise, $(ux_1x_7x_6y_1y_2)$ is an induced 6-cycle. Similarly, u is not adjacent to x_{10} (see $(x_{10}x_9x_8y_3y_2u)$). This completes the proof of Claim 30, since the only remaining possible neighbours of u are x_2 and x_3 , and these do not have a common neighbour. \square

By Claim 30, we reach a contradiction using the usual double counting argument. This completes the proof of Theorem 9. \square

6 The proof of Theorem 3

In this section we shall finish the proof of Theorem 3 by combining Theorem 5 along with some facts we have obtained regarding forbidden substructures in maximal $\{C_3, C_5\}$ -free graphs of large minimum degree. First, we prove the following proposition, which records several useful properties of the graphs F_k that we shall need in the sequel.

Proposition 31. *The following properties of F_k hold.*

1. *Every two distinct vertices in F_k ($k \geq 2$) are contained in a 7-cycle.*
2. *Let x and y be distinct vertices in F_k . Then there is a path of length 1, 3 or 5 between x and y .*
3. *Let F be a copy of F_k in a maximal $\{C_3, C_5\}$ -free graph G with $\delta(G) > n/5$. Then every vertex in G has either $k - 1$ or k neighbours in F .*
4. *Let F be a copy of F_k in a maximal $\{C_3, C_5\}$ -free graph G with $\delta(G) > n/5$. Denote the vertices of F by x_1, \dots, x_{5k-3} and its edges by the pairs $x_i x_j$ for which $|i - j| \equiv 1 \pmod{5}$. Then for every vertex u in G there is a vertex x_i in F such that the neighbours of u in F are the neighbours of x_i in F , except at most one of x_{i-1} and x_{i+1} . In particular, F is well-behaved as a subgraph of G .*

Proof.

Property 1

Denote the vertices and edges of F_k as above (see 4). Property 1 clearly holds for $k = 2$. Now suppose that $k \geq 3$ and let x_i and x_j be two distinct vertices in F_k . Suppose that $i < j$. If $j \leq i + 6$, the two vertices are in the 7-cycle $(x_i \dots x_{i+6})$. Otherwise, x_i and x_j are two vertices in the graph induced by $V(F_k) \setminus \{x_{i+1}, \dots, x_{i+6}\}$ which is a copy of F_{k-1} . Then, by induction, x_i and x_j are in a copy of a 7-cycle in F_k .

Property 2

It is easy to check Property 2 for a 7-cycle. Now by Property 1, any two distinct vertices of F_k are in a 7-cycle in F_k , so we are done.

Property 3

We prove Property 3 by induction on k . For $k = 1$ the result is clear (recall that F_1 is an edge). For $k = 2$, the result easily follows from Theorem 5. So suppose that $k \geq 3$ and the result holds for smaller values of k . Let F be a copy of F_k in G as in the statement of Property 3, denote its vertices and edges as before, and let u be a vertex of G .

Assume first that u has $k + 1$ neighbours in F . If u has at most one neighbour in some consecutive interval x_i, \dots, x_{i+4} of five vertices, then u has at least k neighbours in the copy of F_{k-1} induced on $F \setminus \{x_i, \dots, x_{i+4}\}$, a contradiction to the induction hypothesis. Therefore, u has at least two neighbours in every consecutive interval of five vertices. Suppose, without loss of generality, that u is adjacent to x_1 . Then u has at least $1 + 2(k - 2) \geq k$ neighbours (recall that $k \geq 3$) in the copy of F_{k-1} induced on $F \setminus \{x_{5k-7}, \dots, x_{5k-3}\}$, a contradiction. If u has at most $k - 2$ neighbours in F , one of which is, say, x_1 , then u has at most $k - 3$ neighbours in the copy of F_{k-1} induced on $F \setminus \{x_1, \dots, x_5\}$, contradicting the induction hypothesis. It follows that u has either $k - 1$ or k neighbours in F , as required.

Property 4

Let F and G be as before, and suppose that u has $k - 1$ neighbours in F . Then one may find five consecutive vertices $x_\ell, \dots, x_{\ell+4}$ which are not neighbours of u . Let F' be the copy of F_{k-1} given by $F \setminus \{x_\ell, \dots, x_{\ell+4}\}$. Then by induction there is a vertex x of F' such that u is joined to all neighbours of x in F' . We claim that $x = x_{\ell-1}$ or $x = x_{\ell+5}$. Indeed, note that x must be adjacent to precisely one of $x_{\ell-1}, x_{\ell+5}$ (it cannot be adjacent to both); otherwise, u has no neighbour in the 7-cycle $(x_{\ell-1}x_\ell \dots x_{\ell+5})$, contradicting Theorem 5. Suppose, without loss of generality, that x is joined to $x_{\ell-1}$. Since u must have a neighbour in the 7-cycle $(x_\ell x_{\ell+1} \dots x_{\ell+6})$, u is also adjacent to $x_{\ell+6}$. It follows that $x = x_{\ell+5}$ and u has $k - 1$ neighbours in F which are precisely the neighbours of $x_{\ell+5}$, except for $x_{\ell+4}$.

Now suppose that u has precisely k neighbours in F . Then we may find two neighbours of u that are at distance at most four. We claim that this implies there must be two neighbours at distance two apart. Indeed, they cannot be at distance three (this would produce a 5-cycle). So suppose these neighbours are at distance four and suppose they are x_i and x_{i+4} . Then $(ux_{i+4}x_{i+5}x_{i+6}x_i)$ is a 5-cycle in G , a contradiction. Accordingly, we may assume without loss of generality that u is adjacent to both x_2 and x_{5k-3} . Consider the copy of F_{k-1} given by $F \setminus \{x_3, \dots, x_7\}$ and apply induction. Clearly, we must have u joined to x_7 (u 's only possible neighbour in $\{x_3, \dots, x_7\}$) and the neighbourhood of u in F is precisely the neighbourhood of x_1 in F . This completes the proof of Property 4. \square

We actually prove the following theorem, which clearly implies Theorem 3:

Theorem 32. *Let G be a maximal $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > n/5$. For every integer $k \geq 2$, if G contains no copy of F_k , then G is homomorphic to F_{k-1} .*

Proof. We shall use induction on k . For $k = 2$ we need to show that if G contains no copy of C_7 , then G must be bipartite. Suppose otherwise and let $C = (x_1 \dots x_{2\ell+1})$ be an odd cycle in G of minimal length. Note that, by assumption, $\ell \geq 4$. Also, the minimality of C implies that it is induced. Since the edge x_1x_4 is missing there must be a 4-path connecting x_1 and x_4 (a 2-path is impossible). It follows that there is an odd closed walk of length at most 7 between x_1 and x_4 which contains an odd cycle of length at most 7. Hence G contains a C_7 , contrary to our assumption, and so G must indeed be bipartite.

Now fix $k \geq 3$ and suppose the result holds for smaller values of k . Let G be as in the statement of the theorem and suppose it contains no copy of F_k . If G contains no copy of F_{k-1} , then by induction G is homomorphic to F_{k-2} . But F_{k-1} contains F_{k-2} , so we are done. Hence we may assume that G contains a copy of F_{k-1} . Let H be a vertex-maximal blow-up of F_{k-1} in G with vertex classes X_1, \dots, X_{5k-8} , where the edges of H are $X_i - X_j$ edges for which $|i - j| \equiv 1 \pmod{5}$. Our aim is to show that G is a blow-up of F_{k-1} , or, in other words, that H spans all vertices in G . Note that by Property 3 in Proposition 31, every vertex in $V(G) \setminus V(H)$ has at most $k - 1$ neighbours in F_{k-1} .

Suppose $u \in V(G) \setminus V(H)$ is adjacent to vertices in precisely $k - 1$ of the classes of H . Without loss of generality, by Property 4, we may assume that these classes are those in the neighbourhood of vertices in X_1 , i.e., $X_2, X_7, \dots, X_{5k-8}$, and let $J = \{2, 7, \dots, 5k - 8\}$ be the set of indices j such that u has a neighbour in X_j . We claim that u must be adjacent to every vertex in each of these classes, contradicting the assumption that H is a vertex-maximal blow-up in G . Suppose this is not the case. By Property 3, u has a non-neighbour in at most one of the sets X_j with $j \in J$ (indeed, otherwise we find a copy of F_{k-1} in which u has at most $k - 3$ neighbours). Furthermore, by Property 4, we may assume that this set is X_2 . Let $y \in X_2$ be a neighbour of u and let $z \in X_2$ be a non-neighbour of u .

Owing to the missing edge uz , and by the edge-maximality of G , there must exist a 4-path $uw_1w_2w_3z$ in G between u and z (a 2-path is impossible). Consider the $(5k - 3)$ -cycle $C = (uw_1w_2w_3zx_1x_{5k-8} \dots x_3y)$, where $x_i \in X_i$ (see Figure 18).

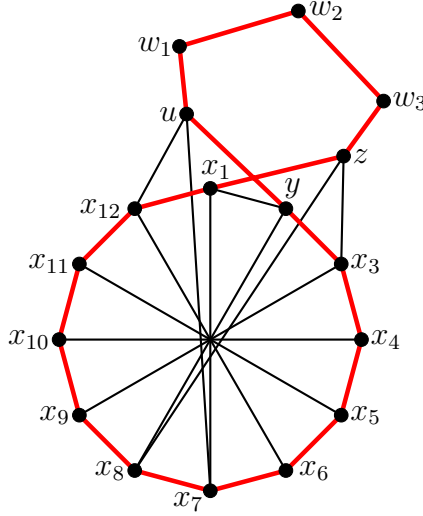


Figure 18: the $(5k - 3)$ -cycle C obtain from u and H

Our aim is to show that $V(C)$ induces a copy of F_k , contrary to our assumption on G . Relabel the cycle C in order as $(z_0z_1 \dots z_{5k-4})$, so that $z_0 = u, z_i = w_i$ for $i = 1, 2, 3$, $z_4 = z, z_5 = x_1$, $z_i = x_{5k-2-i}$ for $6 \leq i \leq 5k - 5$, and $z_{5k-4} = y$. We must check that all chords of lengths $1 + 5t$ for $t = 0, \dots, k - 1$ are present in the graph induced on $V(C)$. Note that all possible chords of these lengths that are not incident with a vertex in $S = \{u, w_1, w_2, w_3\} = \{z_0, z_1, z_2, z_3\}$ are present, since all vertices in $V(C) \setminus S$ are in an appropriate copy of F_{k-1} . So we must check that all possible chords incident with a vertex in S are present. This is summarized in the following claim, where we temporarily revert to the original labelling of C :

Claim 33. *The following hold:*

- $N(u, V(C)) = \{w_1, y\} \cup \{x_{5\ell+2} : 1 \leq \ell \leq k - 2\}$.

- $N(w_1, V(C)) = \{u, w_2\} \cup \{x_{5\ell+1} : 1 \leq \ell \leq k-2\}.$
- $N(w_2, V(C)) = \{w_1, w_3\} \cup \{x_{5\ell} : 1 \leq \ell \leq k-2\}.$
- $N(w_3, V(C)) = \{z, w_2\} \cup \{x_{5\ell-1} : 1 \leq \ell \leq k-2\}.$

Proof. Observe that the first item is immediate from our choice of u . Fix some ℓ with $1 \leq \ell \leq k-2$. Note that every vertex in X_2 is joined to $x_{5(\ell-1)+3} = x_{5\ell-2}$. In particular, y and z are joined to $x_{5\ell-2}$. Consider the 12-cycle $C' = (uw_1w_2w_3zx_{5\ell-2} \dots x_{5\ell+2}x_1y)$, with two consecutive diagonals $yx_{5\ell-2}$ and x_1z . Observe that C' gives rise to another 12-cycle $C'' = (uw_1w_2w_3zx_1x_{5\ell+2}x_{5\ell+1} \dots x_{5\ell-2}y)$ with two consecutive diagonals yx_1 and $ux_{5\ell+2}$. By Theorem 7, either C' or C'' has all of its diagonals present. However, it cannot be C' , since u cannot be adjacent to $x_{5\ell-1}$. Therefore, C'' has all diagonals present: $w_1x_{5\ell+1}$, $w_2x_{5\ell}$, and $w_3x_{5\ell-1}$ are edges in G . This completes the proof of Claim 33. \square

It remains to check that Claim 33 produces chords of the right lengths. We do this for chords incident with w_1 ; the other cases follow identically. Indeed, $w_1 = z_1$ so we must check that z_1 is joined to $z_{1+(1+5t)}$ for $t = 0, 1, \dots, k-1$. This is obviously true for $t = 0$ and $t = k-1$, so let $1 \leq t \leq k-2$. Then the above is equivalent to w_1 being joined to $x_{5k-2-(1+(1+5t))} = x_{5(k-t-1)+1}$, where $1 \leq k-t-1 \leq k-2$, which clearly follows by Claim 33. Accordingly, there is a copy of F_k in G contrary to our assumption, so u must be adjacent to every vertex in X_j for all $j \in J$. But then we may place u in X_1 and produce a blow-up of F_{k-1} of larger order, which is impossible by our choice of H . It follows that every vertex in $V(G) \setminus V(H)$ is adjacent to vertices in at most $k-2$ of the sets X_i . In fact, by Item 4 of Proposition 31, it follows that every vertex in $V(G) \setminus V(H)$ is adjacent to precisely $k-2$ of the X_i 's.

Before proceeding, let us introduce a bit of notation and terminology. Let \tilde{H} be the graph with vertex set $\{X_1, \dots, X_{5k-8}\}$, where an edge X_iX_j is present whenever the pair (X_i, X_j) induces a complete bipartite graph in G . As H is a blow-up of F_{k-1} , \tilde{H} is isomorphic to F_{k-1} . We say that a vertex v is *joined* to a subset $X \subseteq V(G)$ if v is adjacent to every vertex of X .

It follows from Property 4 of Proposition 31 that the sets Y_i , where $i = 1, \dots, 5k-8$, defined below, form a partition of $V(G) \setminus V(H)$ (see Figure 19). Note that each of these sets is independent (as G is triangle-free).

$$Y_i = \{u \in V(G) \setminus V(H) : u \text{ is joined to } X_{i+1}, X_{i+6}, \dots, X_{i+5k-14}\}$$

The following claim asserts that whether or not there are edges between Y_i and Y_j depends on whether or not there are edges between X_i and X_j . We are then able to ‘absorb’ Y_i into X_i , for each i .

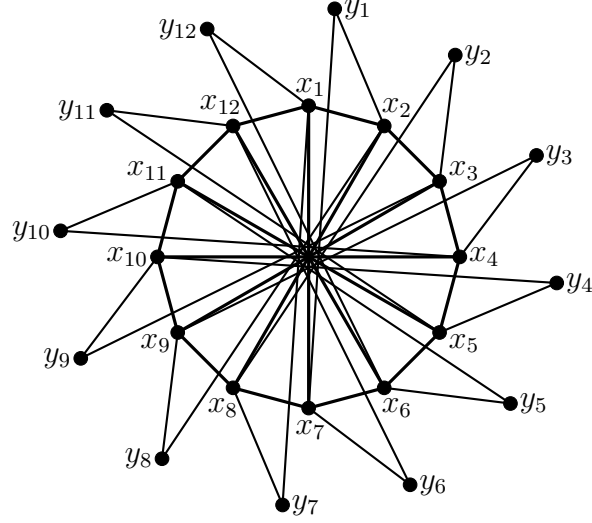


Figure 19: the sets X_i and Y_i

Claim 34. *Let $k \geq 3$ and $1 \leq i, j \leq 5k - 8$. If j is such that $X_j \notin N_{\tilde{H}}(X_i)$, then there are no edges between Y_i and Y_j .*

Proof. Without loss of generality, set $i = 1$. Suppose j is such that $X_j \notin N_{\tilde{H}}(X_1)$. We may assume that $j \neq 1$, as each Y_i is independent. Then $j = 5l + r$, where $l \in \{0, \dots, k - 3\}$ and $r \in \{3, 4, 5, 6\}$. Towards a contradiction, suppose there is an edge $y_1 y_j$ between Y_1 and Y_j . We consider four cases, according to the value of r . Suppose first that $r = 3$. Then we find the following 5-cycle $(y_1 x_2 x_{5l+3} x_{5l+4} y_j)$. If $r = 4$, we find the induced 6-cycle $(y_1 x_2 x_{5l+3} x_{5l+4} x_{5l+5} y_j)$. If $r = 5$, there is, again, an induced 6-cycle $(y_1 x_2 x_1 x_{5k-8} x_{5k-9} y_j)$. Finally, if $r = 6$, there is a 5-cycle $(y_1 x_2 x_{5l+8} x_{5l+7} y_j)$.

For each of the possible values of r , we reached a contradiction by showing that G contains either a 5-cycle or an induced 6-cycle. Claim 34 follows. \square

Let $Z_i = X_i \cup Y_i$. Note that the sets Z_i are independent and they partition $V(G)$. It follows from Claim 34 that there are no $Z_i - Z_j$ edges if $X_i X_j \notin E(\tilde{H})$. By maximality of G , all $Z_i - Z_j$ edges are present if $X_i X_j \in E(\tilde{H})$, implying that G is a blow-up of F_{k-1} . In particular, G is homomorphic to F_{k-1} , as required to complete the proof of Theorem 32. \square

We are then able to establish the following result, as stated in the Introduction, which gives a precise minimum degree condition (depending on k) for forcing a $\{C_3, C_5\}$ -free graph to be homomorphic to F_{k-1} .

Theorem 4. *Let G be a $\{C_3, C_5\}$ -free graph on n vertices with $\delta(G) > \frac{k}{5k-3}n$. Then G is homomorphic to F_{k-1} .*

Proof. Note that we may assume that G is maximal $\{C_3, C_5\}$ -free. By Theorem 32, if G is not homomorphic to F_{k-1} , it contains a copy F of F_k . The number of edges between $V(F)$ and $V(G) \setminus V(F)$ is at most $k(n - (5k - 3))$, since every vertex in G has at most k neighbours in F , by Proposition 31. It follows that there is a vertex u in F with at most $\frac{kn}{5k-3} - k$ neighbours outside of F . Since u has k neighbours in F , it follows that u has degree at most $\frac{kn}{5k-3}$, a contradiction to the minimum degree condition. \square

7 Homomorphism thresholds

Recall that, given a family of graphs \mathcal{H} , the homomorphism threshold $\delta_{\text{hom}}(\mathcal{H})$ of \mathcal{H} is the infimum of d such that every \mathcal{H} -free graph with n vertices and minimum degree at least dn is homomorphic to a bounded \mathcal{H} -free graph. In this section we provide the proof of Theorem 1, which states that the homomorphism threshold of $\{C_3, C_5\}$ is $1/5$. We also prove that $\delta_{\text{hom}}(C_5) \leq 1/5$ by showing that C_5 -free graphs of large enough minimum degree are also triangle-free.

Theorem 1. *The homomorphism threshold of $\{C_3, C_5\}$ is $1/5$.*

Proof. Denote $\delta = \delta_{\text{hom}}(\{C_3, C_5\})$. First, we show that $\delta \geq 1/5$. We note that F_k is not homomorphic to a $\{C_3, C_5\}$ -free graph H with fewer than $|F_k|$ vertices. Indeed, suppose otherwise. Then two vertices x and y in F_k are mapped to the same vertex u in H . By Property 2 there is a path P of length 1, 3 or 5 between x and y . Clearly, P cannot have length 1 (because the set of vertices mapped to the same vertex is independent). It follows that P has length 3 or 5. This implies that the path P is mapped to a cycle of length 3 or 5, a contradiction. It follows that, for each $k \geq 1$, F_k is a $\{C_3, C_5\}$ -free graph with minimum degree at least $|F_k|/5$, which is not homomorphic to a $\{C_3, C_5\}$ -free graph on fewer than $|F_k|$ vertices. Hence, indeed, $\delta \geq 1/5$.

It remains to show that $\delta \leq 1/5$. Let $\varepsilon > 0$ be fixed. Suppose that G is a $\{C_3, C_5\}$ -free on n vertices and minimum degree at least $(1/5 + \varepsilon)n$. Let k be such that $\frac{k}{5k-3} < 1/5 + \varepsilon$. Then, by Theorem 4, G is homomorphic to F_{k-1} . This shows that $\delta \leq 1/5 + \varepsilon$. Since ε was arbitrary, we conclude that $\delta \leq 1/5$. \square

7.1 C_5 -free implies C_3 -free when $\delta(G) > n/6 + 1$

It would be interesting to determine the homomorphism threshold of C_5 . The following lemma enables us to easily obtain an upper bound.

Lemma 35. *Let G be a C_5 -free graph on n vertices with minimum degree larger than $n/6 + 1$. Then, if n is sufficiently large, G is triangle-free.*

Before proving Lemma 35, we use it to prove Corollary 2, which provides an upper bound on the homomorphism threshold $\delta_{\text{hom}}(C_5)$. We currently do not have any nontrivial lower bound on $\delta_{\text{hom}}(C_5)$.

Corollary 2. *The homomorphism threshold of C_5 is at most $1/5$.*

Proof. Suppose that G is a C_5 -free graph on n vertices and minimum degree at least $(1/5 + \varepsilon)n$ for some fixed $\varepsilon > 0$. Then, by Lemma 35, G is also triangle-free. It follows from Theorem 1 that G is homomorphic to a C_5 -free (and C_3 -free) graph H of order at most $C = C(\varepsilon)$. Hence, indeed, $\delta_{\text{hom}}(C_5) \leq 1/5$. \square

We now turn to the proof of Lemma 35.

Proof of Lemma 35. We start by showing that every vertex in G is incident with at most 13 *triangular edges* (i.e. edges on triangles). To see this, suppose that u is incident with at least 14 triangular edges. In other words, the neighbourhood $N(u)$ of u contains edges that span at least 14 vertices. The following claim implies that there is a set X of seven neighbours of u such that every vertex in X has a neighbour in $N(u) \setminus X$.

Claim 36. *Let H be a graph with n vertices and no isolated vertices. Then there is a set X of size at least $n/2$ such that every vertex in X has a neighbour outside of X .*

Proof. We note that it suffices to prove the claim under the assumption that H is connected. Indeed, for each component H_i of H , we may pick a set X_i as in the claim, and let X be the union of the X_i 's. So now we assume that H is connected. Because of the assumption that there are no isolated vertices, we may assume that $|H| \geq 2$.

Let u be a vertex for which $H \setminus \{u\}$ is connected. Let v be a neighbour of u . Consider the graph $H' = H \setminus \{u, v\}$. Let H_1, \dots, H_t be the connected components of H' . We pick a set X_i for each $i \in [t]$ as follows: if H_i consists of a single vertex x_i , then x_i must be adjacent to v , and we take $X_i = \{x_i\}$; otherwise, if H_i has at least two vertices, then by induction there is a set X_i of size at least $|H_i|/2$ such that every vertex in X_i has a neighbour outside of X_i (but in H_i). Let $X = \bigcup_{i=1}^t X_i \cup \{u\}$. It is easy to check that X satisfies the requirements of Claim 36. \square

Let Y be a set of at most seven neighbours of u , which is disjoint from X and satisfies that every vertex in X has a neighbour in Y . Due to the minimum degree condition, we may find two distinct vertices x_1 and x_2 in X that have a common neighbour z outside of $X \cup Y \cup \{u\}$. Let $y \in Y$ be a neighbour of x_1 . Then we find the 5-cycle $(x_1 y u x_2 z)$, a contradiction. Thus, indeed, every vertex is incident with at most 13 triangular edges.

We now show that G contains no two triangles that intersect in an edge.

Claim 37. *G contains no distinct vertices x_1, x_2, x_3, x_4 such that $x_1 x_2 x_3$ and $x_2 x_3 x_4$ are triangles.*

Proof. We note that x_1 and x_4 do not have common neighbours (apart from x_2 and x_3). Indeed, suppose that y is such a common neighbour. Then $(y x_1 x_2 x_3 x_4)$ is a 5-cycle. Similarly, x_1 and x_2 do

not have a common neighbour. By symmetry, the following pairs do not have common neighbours (outside of x_1, x_2, x_3, x_4): $\{x_1, x_3\}$, $\{x_2, x_4\}$ and $\{x_3, x_4\}$. Finally, x_2 and x_3 have at most 13 common neighbours (because every vertex is incident with at most 13 triangular edges). Denote by N_i the set of neighbours of x_i , that are not in $\{x_1, x_2, x_3, x_4\}$ and are not common neighbours of x_2 and x_3 (see Figure 20a).

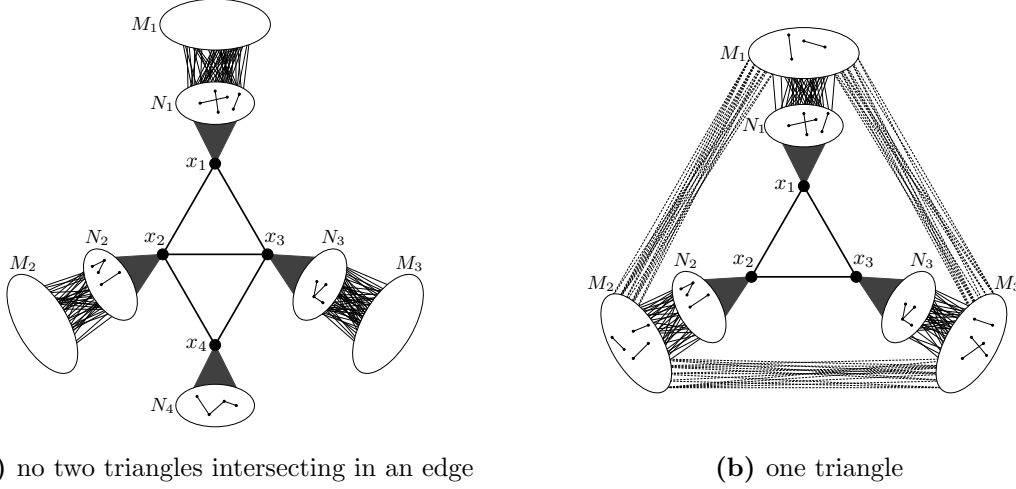


Figure 20: no triangles

The following properties of N_i holds.

- The sets N_i are pairwise disjoint.
- $|N_i| \geq n/6 - 14$ for $i \in [4]$.
- The edges induced by N_i span at most 13 vertices (by Claim 36).
- There are no edges between N_i and N_j for $i \neq j$.

Indeed, if $i = 1$ and $j = 2$ then $(y_1 x_1 x_3 x_2 y_2)$ is a 5-cycle, a contradiction. A contradiction may be reached similarly for all other choices of i and j .

- There are no vertices with neighbours in two of the sets N_1, N_2, N_3 .

Indeed, suppose that z is a neighbour of y_1 and y_2 from N_1 and N_2 . Then $(z y_1 x_1 x_2 y_2)$ is a 5-cycle, a contradiction.

Denote by M_i the set of neighbours of N_i (apart from $N_i \cup \{x_i\}$; see Figure 20a). We note that $|M_i| \geq n/6 - 14$. Indeed, every vertex in N_i has at most 13 neighbours in the neighbourhood of x_i , thus all but 14 of its neighbours are in M_i . The sets $N_1, N_2, N_3, N_4, M_1, M_2, M_3$ are seven pairwise disjoint sets of size at least $n/6 - 14$, a contradiction. \square

Finally, suppose that G contains a triangle $x_1x_2x_3$. By Claim 37, the sets N_i of neighbours of x_i (outside of $\{x_1, x_2, x_3\}$) are disjoint. As in Claim 37, there are no edges between N_i and N_j for $i \neq j$. Similarly, no two of the sets N_1, N_2, N_3 have a common neighbour. Denote by M_i the sets of neighbours of N_i outside of $N_i \cup \{x_i\}$ (see Figure 20b). Note that $|N_i| > n/6 - 1$ and $|M_i| > n/6$ (by Claim 36, there is a vertex u in N_i with no neighbours in N_i ; all of u 's neighbours, apart from x_i , are in M_i). The sets $N_1, N_2, N_3, M_1, M_2, M_3, \{x_1, x_2, x_3\}$ are pairwise disjoint. Thus, their union has size larger than n , a contradiction. It follows that G is triangle-free. \square

We remark that the minimum degree condition in Lemma 35 is best possible, as can be seen by the example depicted in Figure 21.

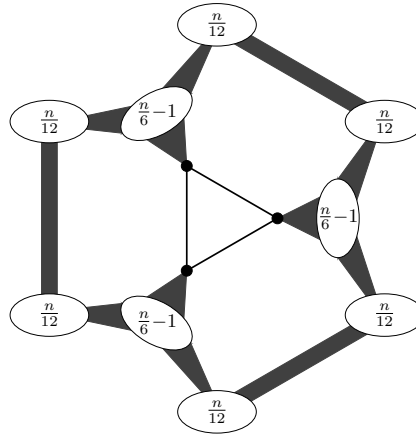


Figure 21: a C_5 -free but not C_3 -free graph with minimum degree $n/6 + 1$

8 Final remarks

We are able to determine precisely the structure of $\{C_3, C_5\}$ -free graphs with high minimum degree, and thereby deduce the value of the homomorphism threshold $\delta_{\text{hom}}(\{C_3, C_5\})$. It would be very interesting to extend this result to $\{C_3, \dots, C_{2\ell-1}\}$ -free graphs. Recall that, for integers $k \geq 2, \ell \geq 3$, $F_{k,\ell}$ is the graph obtained from a $((2\ell - 1)(k - 1) + 2)$ -cycle by adding all chords joining vertices at distances $j(2\ell - 1) + 1$ for $j = 0, 1, \dots, k - 1$. In light of our Theorem 3 it is natural to ask whether or not a $\{C_3, \dots, C_{2\ell-1}\}$ -free graph on n vertices with minimum degree larger than $\frac{n}{2\ell-1}$ is homomorphic to $F_{k,\ell}$ for some k . Rather surprisingly it turns out that this is false when $\ell \geq 4$ is even, as shown by the following construction due to Oliver Ebsen [7].

Suppose that $\ell \geq 4$ is even. Starting with a complete graph on 4 vertices, subdivide two independent edges by an additional $2\ell - 6$ vertices and subdivide the remaining four edges by an additional two vertices each. Denote the resulting graph by T_ℓ . It is easy to check that this graph is maximal $\{C_3, \dots, C_{2\ell-1}\}$ -free. To obtain large minimum degree assign weight 2 to each vertex of the original

K_4 and to $\ell-4$ vertices of the ‘long’ subdivided edges, and assign weight 1 to the remaining vertices. This may be done in such a way that each vertex has weight 3 in its neighbourhood (as ℓ is even). To obtain an unweighted graph of order n simply blow up each vertex with an independent set of size proportional to its weight. Then the resulting graph T_ℓ^* is maximal $\{C_3, \dots, C_{2\ell-1}\}$ -free and $\delta(T_\ell^*) = \frac{3n}{6\ell-4} > \frac{n}{2\ell-1}$. However, we claim that T_ℓ is not homomorphic to $F_{k,\ell}$, for any k (and therefore no blow-up of T_ℓ is homomorphic to any $F_{k,\ell}$). Indeed, suppose otherwise and let k_0 be minimal such that T_ℓ is homomorphic to $F_{k_0,\ell}$ (obviously $k_0 \geq 2$). It is easy to check that, for $k \geq 2$ and any vertex v , $F_{k,\ell} - \{v\}$ is homomorphic to $F_{k-1,\ell}$. Since T_ℓ is not homomorphic to $F_{k_0-1,\ell}$, it follows that T_ℓ must be a blow-up of $F_{k_0,\ell}$ with all parts nonempty. Since T_ℓ has precisely 4ℓ vertices this implies that $k_0 \leq 3$, and so either T_ℓ is a blow-up of $F_{2,\ell} = C_{2\ell+1}$ or of $F_{3,\ell}$. But no two vertices of T_ℓ have the same neighbourhood so the former case is impossible. Moreover, $F_{3,\ell}$ has exactly 4ℓ vertices as well, but clearly T_ℓ is not isomorphic to $F_{3,\ell}$ (as T_ℓ is not regular, for example). It follows that T_ℓ is not homomorphic to any $F_{k,\ell}$ as claimed.

We do not know whether Theorem 3 extends naturally to $\{C_3, \dots, C_{2\ell-1}\}$ -free graphs when $\ell \geq 5$ is odd, and it would be interesting to pursue this line of research further.

Recall that the homomorphism threshold of a family of graphs \mathcal{H} is the infimum of d satisfying that every \mathcal{H} -free graph with n vertices and minimum degree at least dn is homomorphic to an \mathcal{H} -free graph of bounded order (depending on d but not on n). Despite the above remarks concerning the extension of Theorem 3 to general odd-girth graphs, we still make the following conjecture concerning the homomorphism threshold of $\{C_3, \dots, C_{2\ell-1}\}$ -free graphs for $\ell \geq 4$.

Conjecture 38. *Let $\ell \geq 4$ be an integer. Then $\delta_{\text{hom}}(\{C_3, \dots, C_{2\ell-1}\}) = \frac{1}{2\ell-1}$.*

We have also obtained an upper bound on $\delta_{\text{hom}}(C_5)$, namely, that it is at most $1/5$. We ask if it is true that $1/5$ is the correct value.

Question 39. *Is it true that $\delta_{\text{hom}}(C_5) = 1/5$?*

In fact, any nontrivial (i.e. non-zero) lower bound on $\delta_{\text{hom}}(C_5)$ would be interesting. In order to obtain such a lower bound, one would have to find, in particular, a family of graphs that have large minimum degree, are C_5 -free and are not 4-colourable (indeed, otherwise, the graphs are homomorphic to K_4 , which is clearly C_5 -free). Although it is well known that such graphs exist, it seems hard to find explicit examples, especially with the added condition that they are not homomorphic to C_5 -free graphs of bounded order.

Finally, it would be very interesting to determine the value of $\delta_{\text{hom}}(C_{2\ell-1})$ in general.

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